Differentials in Exile

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Feynman Quote

“ The trouble with mathematicians is they prefer precision to clarity”.
Richard Feynman (?).

Some quick questions

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A ladder is against a wall, 3 ft up the wall and 4 feet along the ground. if you move the top of the ladder down 6 inches, how much does the bottom move out?

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Note there is no time involved, no rates, no "per" anything.
CHANGE not RATE OF CHANGE.

Rules for the “d” operator

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Rule Times $d(uv) = vdu + udv$
Rule Derivative $df(x) = f'(x)dx$.

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$$dx = -\frac{3}{4}(-6) = 4.5 \text{ inches.}$$

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Since $V = 4\pi r^3/3$, relative error for V is three times relative error for r. We need 2% accuracy for r.

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Measurement strand of NCTM standards.

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Nonstandard analysis... come on!! The issue is not dx, it's $=$ and local linearity.

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Ask any undergraduate physics major about the sphere problem and they won't hesitate, plus they will give you a lovely argument with differentials.

Best example, Lagrange multipliers and economics

The production problem;

$$\text{Maximize Production: } P(K, L) = K^{0.3}L^{0.7}$$

$$\text{Budget constraint: } B(K, L) = 10K + 40L = 200$$

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We have two homogeneous, linear equations in dK and dL. Lagrange "By the theory of equations" the first must be a multiple of the second, $dP = \lambda dB$, if we are to have nontrivial solutions.

Lagrange continued

We have then $dP/dK = \lambda dB/dK$, $dP/dL = \lambda dB/dL$. Solving each for λ,and simplifying, we get:

$$\frac{K}{L} = \frac{0.7/10}{0.3/40}.$$

Makes sense. Note this rule doesn't depend on the budget you have.

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And $1/\lambda$ is the marginal cost of producing an extra unit.

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$$\frac{dy}{y} = \frac{f'(x)dx}{y} = \frac{f'(x)}{y/x}\frac{dx}{x}.$$

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For example, suppose you know x to three significant digits, $x = 314 \approx 100\pi$. How many significant digits do you know for $sinx$?
Answer: none. Conditioning number is $1 \cdot 314/(-0.15) \approx -2000$. You lose all significant digits.