Teaching Probability and Decision Making Theory Through the Game of Squash

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The Game of Squash

- Squash is played either one vs one or two vs two in an indoor court
- Game played with racquets and hallow, rubber balls
- The court is a confined space
- The game are either best of 3 or best of 5

Squash Scoring Rules

- Each match is played to 9 points
  - The goal is to be the first player to earn 9 points
  - The match ends at 9 points unless there is a tie at 8 points
- Only the player serving can earn points
- If the non-serving player wins the volley, they are to receive no points, but they get the serve
- A let is when no player earns points and the server serves again

Scoring if there is a tie

- The player who has earned 8 points first gets to make the decision to call either “Set 1” or “Set 2”
- Set 1: the next player who earns 1 point wins
- Set 2: the player who earns wins by 2 points

Variables

- x = P(A) = probability that Player A wins point
- y = P(B) = probability that Player B wins point
- z = $P(B_9)$ = probability that Player B wins Set One scenario*
- w = $P(B_{10})$ = probability that Player B wins Set Two scenario*

  - *Assuming that Player B reaches 8 points before Player A

Geometric Series

- Recall, a series is a sum of a set of terms where the previous term is multiplied by the same number, r, to find the next term
- r is the common ratio
- a or a_1 are the first term

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_\infty = \frac{a_1}{1 - r}$$

$$P(B_9) = P(A)^0 P(B)^2 + P(A)^1 P(B)^3 + \ldots$$

$$= P(B)^2 \sum_{n=0}^{\infty} \left[P(A) P(B) \right]^n = \frac{P(B)^2}{1 - P(A) P(B)}$$

Three Cases to find P(B_{10})

- Case I: Player B reaches both 9 and 10 points before Player A reaches 9 points
- Case II: Both players reach 9 points, with Player A serving
- Case III: Both players reach 9 points, with Player B serving

$$P\left(B_{10}\right) = \left(\text{Case 1}\right) + \left(\text{Case 2}\right) + \left(\text{Case 3}\right)$$

$$P\left(B_{10}\right)=\left(\text{Case 1}\right)+\left(\text{Case 2}\right)+\left(\text{Case 3}\right)$$

$$\left(\frac{P(B_9)P(B)}{1-P(A)P(B)}\right)+\left(\frac{P(B_9)^2P(A)^2}{1-P(A)P(B)}\right)+\left(\left[1-P(B)\frac{P(B_9)P(B)}{1-P(A)P(B)}\right]\right)$$

$$=\frac{P(B_9)\left[P(B)+P(B_9)P(A)^2\right]+\left[1-P(B_9)\right]P(B_9)P(B)}{1-P(A)P(B)}$$

Then, we can say that

$$z = \frac{y^2}{1 - xy}$$

$$w = \frac{z\left[y + zx^2\right] + \left[1 - z\right]zy}{1 - xy}$$

Since P(B) = 1 – P(A), we may replace *y* with 1 – *x*

$$z = \frac{(1-x)^2}{1-x(1-x)}$$

$$w = \frac{z\left[(1-x)+zx^2\right]+z(1-z)(1-x)}{1-x(1-x)}$$

We may replace *z* in the second equation with $\dfrac{(1-x)^2}{1-x(1-x)}$

Which gives us:

$$w=\frac{\dfrac{(1-x)^2}{1-x(1-x)}\left[(1-x)+\dfrac{(1-x)^2}{1-x(1-x)}x^2\right]+\dfrac{(1-x)^2}{1-x(1-x)}\left[1-\dfrac{(1-x)^2}{1-x(1-x)}\right](1-x)}{1-x(1-x)}$$

Simplifying both formulas give us:

$$z = \frac{(1 - x)^2}{x^2 - x + 1}$$

$$w = \frac{-(1 - x)^3 (x^3 - 2x^2 - 1)}{(x^2 - x + 1)^3}$$

If we set z = w, we are left with

- Recall, z = P(B_9) and w = P(B_{10})

$$\frac{(1-x)^2}{x^2-x+1} = \frac{-(1-x)^3(x^3-2x^2-1)}{(x^2-x+1)^3}$$

$$1=\frac{-\left(1-x\right)\left(x^{3}-2x^{2}-1\right)}{\left(x^{2}-x+1\right)^{2}}$$

$$\left(x^{2}-x+1\right)^{2}+\left(1-x\right)\left(x^{3}-2x^{2}-1\right)=0$$

$$x^{4}-2x^{3}+3x^{2}-2x+1-x^{4}+3x^{3}-2x^{2}+x-1=0$$

$$x^{3}+x^{2}-x=0$$

$$x\left(x^{2}+x-1\right)=0$$

$$x=1,\ 0,\ \ -\frac{1}{2}+\frac{\sqrt{5}}{2}$$

Solutions

- We have found that either A wins 100% of the time and B wins 0%
- OR, A wins 0% of the time and B wins 100%
- OR, the remaining solution: $-\frac{1}{2}+\frac{\sqrt{5}}{2}$ is the Golden Ratio » 0.618.....